Global Exponential Stability For Lotka-Volterra Population Model With Time Varying Delays

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***Abstract*—The global exponential stability for a competitive Lotka-Volterra population model with time varying delays is investigated. A novel exponential stability criterion for the system is derived using the Lyapunov method. These stability conditions are formulated as linear matrix inequalities (LMIs) which can be easily solved by various convex optimization algorithms. An example is given to illustrate the usefulness of our proposed method.**

***Keywords*— Global exponential stability, Lotka-Volterra system, Linear Matrix Inequality, Time-varying delay**

2000 AMS Subject Classification — 34K20, 34K50, 92D25

I. Introduction

The problem of delayed systems has been investigated over the years. The phenomena of time delay are very often encountered in different technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. A delay in the control or state evolution laws may cause an undesirable system transient response or even instability. The analysis of stability for this class of system has been of interest to many researchers.

Nonlinear systems are very often studied in terms of simple mathematical models. The Lotka-Volterra equations provide such a model and have been used to study physical, chemical, ecological and social systems [12]. Stability analysis of population models governed by delay differential equation has been extensively studied in a number of papers (see [3]-[6] and the references there in).

Recently, Zhen and Ma [14] derived a very reasonable condition for the local stability of the competitive Lotka-Volterra system. Park [11] and Sun [13] showed that the estimate on the length of delays in [14] is somewhat conservative and gave a less conservative LMI conditions to guarantee the local stability of the competitive Lotka-Volterra system. However, these results are only concerned with the asymptotic stability, without providing any conditions for exponential stability and any information about the decay rates. It is very well known that for real time computations the fast convergence of solution of a system is essential. Therefore the exponential convergence rate is used to determine the speed of computations. Thus, in general it is important both theoretically and practically to determine the exponential stability and to estimate the exponential convergence rate. Considering this, many researchers have studied the exponential stability analysis for systems with time delays in the literatures [7, 8, 9]. The choice of an appropriate Lyapunov-Krasovskii functional is the key point for deriving of stability criteria. It is known that the general form of this functional leads to a complicated system of partial differential equations (see [10]). That is why many authors considered special form of Lyapunov-Krasovskii functional and thus derived simpler (but more conservative) sufficient conditions.

In this paper, we extend the recent results [11, 13, 14] for the exponential stability and estimate the exponential convergence rates of Lotka - Volterra system with time-varying delays. To the best of author's knowledge, the issue of exponential stability for a competitive Lotka-Volterra population model with time varying delays using LMI approach is remains open, which motivates this paper. Based on Linear Matrix Inequality (LMI), we establish a new LMI condition by using the Lyapunov-Krasovskii functional and applying the Jensen's inequality [4] together with the zero function to guarantee the exponential stability of the competitive Lotka-Volterra system . The obtained stability criterion remains less conservative than conditions discussed in [11], [13] and [14]. Particularly, the maximal allowable length of delays is obtained from LMI and the validity of this result is checked numerically using the effective LMI control toolbox in MATLAB.

Throughout this paper, the notation * represents the elements below the main diagonal of a symmetric matrix. A^T means the transpose of A. We say $X > Y$ if $X - Y$ is positive definite, where X and Y are symmetric matrices of same dimensions. P·P refers to the Euclidean norm for vectors.

II. Main Results

In this section, we derive the necessary and sufficient conditions for exponential stability of the following Lotka-Volterra type competitive system,

$$\dot{x}_1(t) = x_1(t)[b_1 - a_{11}x_1(t-\tau_{11}(t)) - a_{12}x_2(t-\tau_{12}(t))]$$
$$\dot{x}_2(t) = x_2(t)[b_1 - a_{21}x_1(t-\tau_{21}(t)) - a_{22}x_2(t-\tau_{22}(t))] \tag{1}$$

Where $x_1(t)$ and $x_2(t)$ stands for densities of the population at time t. b_i, a_{ij} are positive constants and τ_{ij} denotes the time varying delay.

The initial condition of (1) is given as

$$x_1(s) = \varphi_1(s) \geq 0, -h \leq s \leq 0; \varphi_1(0) > 0,$$
$$x_2(s) = \varphi_2(s) \geq 0, -h \leq s \leq 0; \varphi_2(0) > 0,$$

and let h=max $\{\tau_{ij}\}$.

Under the conditions (see [14])

$$\frac{a_{11}}{a_{21}} > \frac{b_1}{b_2} > \frac{a_{12}}{a_{22}}$$

all positive solutions $x(t) = (x_1(t), x_2(t))$ of system (1) have unique positive equilibrium $x^* = (x_1^*, x_2^*)$:

$$x_1^* = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}}, \qquad x_2^* = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{21}a_{12}}.$$

Let

$$u(t) = x_1(t) - x_1^*, v(t) = x_2(t) - x_2^*,$$

where x_1^* and x_2^* are defined by (2), then the system (1) reduces to the following system

$$\dot{u}(t) = [u(t) + x_1^*][-a_{11}u(t - \tau_{11}(t)) - a_{12}v(t - \tau_{12}(t))]$$
$$\dot{v}(t) = [v(t) + x_2^*][-a_{21}u(t - \tau_{21}(t)) - a_{22}v(t - \tau_{22}(t))].$$

One can see from (3) that the variational system of (1) with respect to the positive equilibrium x^* is given by the linearized system of (3) that is,

$$\dot{u}(t) = -a_{11}x_1^* u(t - \tau_{11}(t)) - a_{12}x_1^* v(t - \tau_{12}(t))$$
$$\dot{v}(t) = -a_{21}x_2^* u(t - \tau_{21}(t)) - a_{22}x_2^* v(t - \tau_{22}(t)).$$

Rewrite (4) in the following matrix form,

$$\dot{x}(t) = \begin{array}{l} -A_{11}x(t - \tau_{11}(t)) - A_{12}x(t - \tau_{12}(t)) \\ -A_{21}x(t - \tau_{21}(t)) - A_{22}x(t - \tau_{22}(t)) \end{array} \qquad (5)$$

where $x^T(t) = (u^T(t), v^T(t))$ is a vector,

$$A_{11} = \begin{bmatrix} a_{11}x_1^* & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & a_{12}x_1^* \\ 0 & 0 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 0 & 0 \\ a_{21}x_2^* & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & 0 \\ 0 & a_{22}x_2^* \end{bmatrix}.$$

The equivalent form of (5) is given by

$$\dot{x}(t) = \begin{array}{l} -Ax(t) + A_{11}[x(t) - x(t - \tau_{11}(t))] \\ + A_{12}[x(t) - x(t - \tau_{12}(t))] \\ + A_{21}[x(t) - x(t - \tau_{21}(t))] \\ + A_{22}[x(t) - x(t - \tau_{22}(t))], \end{array} \qquad (6)$$

where $A = A_{11} + A_{12} + A_{21} + A_{22}$. The equation (6) can be written as

(2)

$$\dot{x}(t) = -Ax(t) + \sum_{i,j=1}^{2} A_{ij}[x(t) - x(t - \tau_{ij}(t))]. \qquad (7)$$

To establish the exponential stability of the positive equilibrium for system (1), it is sufficient to study the exponential stability of system (7). The following lemma will be used in the crucial role for proving our main results.

Lemma 2.1 (Jensen's inequality [4]). For any constant matrix $M \in R^{m \times m}, M = M^T > 0$, scalar $\gamma > 0$, vector (3) function $\omega : [0, \gamma] \to R^m$ such that the integrations concerned are well defined, then

$$-\gamma \int_0^{\gamma} \omega^T(\beta) M \omega(\beta) d\beta$$
$$\leq -(\int_0^{\gamma} \omega(\beta) d\beta)^T M (\int_0^{\gamma} \omega(\beta) d\beta). \qquad (4)$$

Theorem 2.2 *Let the assumption*

$0 \leq \tau_{ij}(t) \leq h_{ij}, \dot{\tau}_{ij}(t) \leq d_{ij}$ with $d_{ij} \geq 0$ and $h_{ij} \geq 0, i, j = 1,2, t \geq 0$,

holds, For given scalars τ_{ij} for $i, j = 1,2$ system (7) is globally exponentially stable, if there exist positive definite matrices $P > 0, Q_{ij} > 0, R_{ij} > 0$ for $i, j = 1,2$ and $S > 0$ such that,

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & -A^T S \\ * & -\Xi_{22} & 0 & 0 & 0 & A_{11}^T S \\ * & * & -\Xi_{33} & 0 & 0 & A_{12}^T S \\ * & * & * & -\Xi_{44} & 0 & A_{21}^T S \\ * & * & * & * & -\Xi_{55} & A_{22}^T S \\ * & * & * & * & * & \Xi_{66} \end{bmatrix} < 0 \quad (8)$$

where

$$\Xi_{11} = -A^T P - PA + 2\alpha P + \sum_{i,j=1}^{2} Q_{ij} - \sum_{i,j=1}^{2} e^{-2\alpha\tau_{ij}(t)}(1-d_{ij})Q_{ij},$$

$$\Xi_{12} = PA_{11} + (1-d_{11})e^{-2\alpha\tau_{11}}Q_{11}, \quad \Xi_{13} = PA_{12} + (1-d_{12})e^{-2\alpha\tau_{12}}Q_{12},$$

$$\Xi_{14} = PA_{21} + (1-d_{21})e^{-2\alpha\tau_{21}}Q_{21}, \quad \Xi_{15} = PA_{22} + (1-d_{22})e^{-2\alpha\tau_{22}}Q_{22},$$

$$\Xi_{22} = (1-d_{11})e^{-2\alpha\tau_{11}}Q_{11} + \frac{1}{\tau_{11}} e^{-2\alpha\tau_{11}}R_{11}, \quad \Xi_{33} = (1-d_{12})e^{-2\alpha\tau_{12}}Q_{12} + \frac{1}{\tau_{12}} e^{-2\alpha\tau_{12}}R_{12},$$

$$\Xi_{44} = (1-d_{21})e^{-2\alpha\tau_{21}}Q_{21} + \frac{1}{\tau_{21}} e^{-2\alpha\tau_{21}}R_{21}, \quad \Xi_{55} = (1-d_{22})e^{-2\alpha\tau_{22}}Q_{22} + \frac{1}{\tau_{22}} e^{-2\alpha\tau_{22}}R_{22},$$

$$\Xi_{66} = \sum_{i,j=1}^{2} \tau_{ij} R_{ij} - \sum_{i,j=1}^{2} (S^T + S).$$

Proof: Considering the Lyapunov-Krasovskii functional as

$$V(t) = V_1 + V_2 + V_3$$

where

$$V_1 = e^{2\alpha t} x^T(t) P x(t) \quad (9)$$

$$V_2 = \sum_{i,j=1}^{2} \int_{t-\tau_{ij}(t)}^{t} e^{2\alpha s} x^T(s) Q_{ij} x(s) ds, \quad (10)$$

$$V_3 = \sum_{i,j=1}^{2} \int_{-\tau_{ij}(t)}^{0} \int_{t+\beta}^{t} e^{2\alpha s} \dot{x}^T(s) R_{ij} \dot{x}(s) ds d\beta, \quad (11)$$

$P > 0, Q_{ij} > 0,$ and $R_{ij} > 0$ are matrices of appropriate dimensions to be determined.

The time derivative of V along the trajectories of (7) is given by

$$\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3.$$

From $(9)-(11)$ we have,

$$\dot{V}_1 = 2\alpha e^{2\alpha t} x^T(t) P x(t) + 2e^{2\alpha t} x^T(t) P \dot{x}(t)$$

$$\leq e^{2\alpha t}[2\alpha\, x^T(t)Px(t)+2x^T(t)P(-Ax(t)+\sum_{i,j=1}^{2}A_{ij}(x(t)-x(t-\tau_{ij}(t)))]$$

$$\dot{V}_2 \leq e^{2\alpha t}[x^T(t)\sum_{i,j=1}^{2}Q_{ij}x(t)-\sum_{i,j=1}^{2}(1-\dot{\tau}_{ij})e^{-2\alpha\tau_{ij}(t)}x^T(t-\tau_{ij}(t))\,Q_{ij}x(t-\tau_{ij}(t))]$$

$$\leq e^{2\alpha t}[x^T(t)\sum_{i,j=1}^{2}Q_{ij}x(t)-\sum_{i,j=1}^{2}(1-d_{ij})e^{-2\alpha\tau_{ij}(t)}x^T(t-\tau_{ij}(t))\,Q_{ij}x(t-\tau_{ij}(t))]$$

$$\dot{V}_3 \leq e^{2\alpha t}[\dot{x}^T(t)\sum_{i,j=1}^{2}\tau_{ij}(t)R_{ij}\;\dot{x}(t)-\sum_{i,j=1}^{2}\int_{t-\tau_{ij}(t)}^{t}e^{2\alpha(s-t)}\;\dot{x}^T(s)\;R_{ij}\dot{x}(s)ds].$$

Using the Lemma 2.1 we have,

$$-\sum_{i,j=1}^{2}\int_{t-\tau_{ij}(t)}^{t}e^{2\alpha(s-t)}\dot{x}^T(s)R_{ij}\;\dot{x}(s)ds$$

$$\leq -e^{2\alpha(s-t)}\sum_{i,j=1}^{2}(\int_{t-\tau_{ij}(t)}^{t}\dot{x}(s)ds)^T(\frac{1}{\tau_{ij}})R_{ij}(\int_{t-\tau_{ij}(t)}^{t}\dot{x}(s)ds).$$

By the fact that, $\int_{t-h(t)}^{t}\dot{x}(s)ds = x(t)-x(t-h(t))$ and for any scalar $s\in[t-h,t]$ we have

$e^{-2\alpha h}\leq e^{2\alpha(s-t)}\leq 1$ further,

$$\dot{V}_3 \leq e^{2\alpha t}[\dot{x}^T(t)\sum_{i,j=1}^{2}\tau_{ij}(t)R_{ij}\;\dot{x}(t)-\sum_{i,j=1}^{2}\int_{t-\tau_{ij}(t)}^{t}e^{-2\alpha\tau_{ij}(t)}[x(t)-x(t-\tau_{ij}(t)]^T$$

$$\times(\frac{1}{\tau_{ij}})R_{ij}[x(t)-x(t-\tau_{ij}(t)]].$$

From (7) for positive matrix $S>0$ we have

$$-\dot{x}^T(t)(S^T+S)\dot{x}(t)+\dot{x}^T(t)S^T\{-Ax(t)+\sum_{i,j=1}^{2}A_{ij}[x(t)-x(t-\tau_{ij}(t)]\}$$

$$+\{-Ax(t)+\sum_{i,j=1}^{2}A_{ij}[x(t)-x(t-\tau_{ij}(t)]\}^T S\dot{x}(t)=0.$$

Set $x_{ij}(t)=x(t)-x(t-\tau_{ij}(t)),$ then

$$\dot{V}(t)=e^{2\alpha t}\{x^T(t)[2\alpha P-A^TP-PA+\sum_{i,j=1}^{2}Q_{ij}-\sum_{i,j=1}^{2}(1-d_{ij})e^{-2\alpha\tau_{ij}(t)}Q_{ij}]x(t)$$

$$+2x^T(t)P\sum_{i,j=1}^{2}A_{ij}x_{ij}(t)+2x^T(t)P\sum_{i,j=1}^{2}(1-d_{ij})e^{-2\alpha\tau_{ij}(t)}Q_{ij}x_{ij}(t)$$

$$-\sum_{i,j=1}^{2}e^{-2\alpha\tau_{ij}(t)}x_{ij}^T(t)(1-d_{ij})Q_{ij}x_{ij}(t)+\dot{x}^T(t)\sum_{i,j=1}^{2}\tau_{ij}(t)R_{ij}\dot{x}(t)-\dot{x}^T(t)(S^T+S)\dot{x}(t)$$

$$-\sum_{i,j=1}^{2} e^{-2\alpha\tau_{ij}(t)} x_{ij}^T(t)(\frac{1}{\tau_{ij}})R_{ij}x_{ij}(t) - x^T(t)A^T\ S\ \dot{x}(t) - \dot{x}^T(t)\ S^T\ Ax(t)$$

$$+\sum_{i,j=1}^{2} x_{ij}^T(t)A_{ij}^T\ S\ \dot{x}(t) + \dot{x}^T(t)\ S^T \sum_{i,j=1}^{2} A_{ij}x_{ij}\}.$$

Thus,

$$\dot{V}(t) \le e^{2\alpha t}\ \xi^T\ \Xi\ \xi$$

where $\xi^T = [x^T(t)\ x_{11}^T(t)\ x_{12}^T(t)\ x_{21}^T(t) x_{22}^T(t)\ \dot{x}^T(t)]$. According to the Theorem 9.8.1 in [5], we conclude that if matrix inequality (8) holds, then system (7) is asymptotically stable. This guarantees the exponential stability with decay rate α of system (7).

For the case $\tau_{ij}(t) = \tau(t)$, $i, j = 1,2$ or certain $\tau_{ij}(t) = 0$, similar to the proof of Theorem 2.2, simplified conditions for the exponential stability of the positive equilibrium of system (7) can be derived from Theorem 2.2 easily. Here we only consider the case $\tau_{ij}(t) = \tau(t)$. The following Corollary is obvious.

Corollary 2.3 Let assumption (A) holds for $\tau(t)$. If there exists positive definite matrices $P > 0, Q_{ij} > 0, R_{ij} > 0$ and $S > 0$ such that

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_{15} & -A^T\ S \\ * & -\Sigma_{22} & 0 & 0 & 0 & A_{11}^T S \\ * & * & -\Sigma_{33} & 0 & 0 & A_{12}^T S \\ * & * & * & -\Sigma_{44} & 0 & A_{21}^T S \\ * & * & * & * & -\Sigma_{55} & A_{22}^T S \\ * & * & * & * & * & \Sigma_{66} \end{bmatrix} < 0$$

where

$$\Sigma_{11} = -A^T P - PA + 2\alpha P + Q - (1-d)e^{-2\alpha\tau(t)} \sum_{i,j=1}^{2} Q_{ij},$$

$$\Sigma_{12} = PA_{11} + (1-d)e^{-2\alpha\tau} Q_{11}, \quad \Sigma_{13} = PA_{12} + (1-d)e^{-2\alpha\tau} Q_{12},$$

$$\Sigma_{14} = PA_{21} + (1-d)e^{-2\alpha\tau} Q_{21}, \quad \Sigma_{15} = PA_{22} + (1-d)e^{-2\alpha\tau} Q_{22},$$

$$\Sigma_{22} = (1-d)e^{-2\alpha\tau} Q_{11} + \tau^{-1}e^{-2\alpha\tau} R_{11}, \quad \Sigma_{33} = (1-d)e^{-2\alpha\tau} Q_{12} + \tau^{-1}e^{-2\alpha\tau} R_{12},$$

$$\Sigma_{44} = (1-d)e^{-2\alpha\tau} Q_{21} + \tau^{-1}e^{-2\alpha\tau} R_{21}, \quad \Sigma_{55} = (1-d)e^{-2\alpha\tau} Q_{22} + \tau^{-1}e^{-2\alpha\tau} R_{22},$$

$$\Sigma_{66} = \tau R - (S^T + S),$$

then the system (7) is exponentially stable

III. Numerical Example

Consider the following system [14]

$$\dot{x}(t) = x(t)[1 - x(t - \tau_{11}(t)) - 0.5\ y(t - \tau_{12}(t))]$$
$$\dot{y}(t) = y(t)[1 - 0.5\ x(t - \tau_{21}(t)) - y(t - \tau_{22}(t))]$$

For convergence rate $\alpha = 0.5, d = 0.5$ and $h = 1.7038$ the LMI solutions of Corollary 2.3 for this example are,

$$P = 10^{-10}\begin{bmatrix} 0.3953 & -0.0013 \\ -0.0013 & 0.4166 \end{bmatrix},$$

$$Q = 10^{-9}\begin{bmatrix} 0.2994 & -0.0104 \\ -0.0104 & 0.3550 \end{bmatrix},$$

$$R = 10^{-9}\begin{bmatrix} 0.7634 & -0.3199 \\ -0.3199 & 0.8004 \end{bmatrix},$$

$$S = 10^{-8}\begin{bmatrix} 0.3187 & -0.1398 \\ -0.1398 & 0.3330 \end{bmatrix}.$$

Applying Theorem 2.2 and LMI Toolbox in MATLAB, we find that the system (19) is exponentially stable. The feasible solutions with convergence rate $\alpha = 0.5$, $\tau = 2,\ d = 0.1$ as follows:

$$P = 10^{-12}\begin{bmatrix} 0.1251 & 0.0069 \\ 0.0069 & 0.1215 \end{bmatrix},$$

$$Q = 10^{-10}\begin{bmatrix} 0.1536 & 0.0065 \\ 0.0065 & 0.1501 \end{bmatrix},$$

$$R = \begin{bmatrix} 0.6774 & -0.0001 \\ -0.0001 & 0.6775 \end{bmatrix},$$

$$S = \begin{bmatrix} 1.9061 & 0.0002 \\ 0.0002 & 1.9060 \end{bmatrix}.$$

TABLE I. COMPARISION AMONG VARIOUS STABILITY CRITERIA

Method	d=0	d=0.1	d=0.5	d=0.9
Zhen and Ma [14]	0.34	-	-	-
Park [11]	0.88	-	-	-
Sun [13]	1.3623	1.3497	1.2071	1.0486
Our result (Corollary 2.3)	1.7282	1.7248	1.7038	1.6275

CONCLUSION

A novel exponential stability criteria for a competitive Lotka-Volterra population model with time varying delays has been provided. The new sufficient criterion has been presented in terms of linear matrix inequalities (LMIs). The results are obtained based on the Lyapunov theory in combination with generalized eigen value problem (GEVP). The validity of the approach has been demonstrated by numerical example. The maximumum allowable delay is compared with the existing results. The criterion is less conservative than those given in the earlier references on the stability for a competitive Lotka-Volterra population model with time varying delays and the comparision result has been shown in Table I.

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