A Critical Study of Compactness Properties of Topological Groups

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Abstract
Topological groups is logically the combination of groups and topological spaces, they are groups and topological spaces at the same time, such that the continuity condition for the group operations connects these two structures together and consequently they are not independent from each other. The generalised structure which includes several fields such pointset topology, algebraic topology, and differential topology. Dubois, D[5],J. Zhang[10], Felix Hausdorff, Maurice Frechet, and Henri Poincare studied its basic properties based on Topological space and some geometrical structure.A pre-topological group G is a group which is also a pre-topological space such that the multiplication mapping of G × G into G sending $x \times y$ into $x \cdot y$, and the inverse mapping of H into G sending x into x^{-1}, are pre-continuous mappings.

Keywords: (Topological groups, Euclidean topology, Hausdorff, algebraic topology, Arbitrary unions and homeomorphism)

1 Introduction

It mainly studied the generic questions in topological algebra is how the relationship between topological properties depend on the underlying algebraic structure. As we all known, a topological group, that is, a group G is endowed with a topology such that the binary operation G × G → G is jointly continuous and the inverse mapping. In : G → G, i.e,. $x \to x^{-1}$, is also continuous. The properties of topological groups have been widely used in the study of topology, analysis and category

Let(X, τ) be a topological space. We say(X, τ) *has the* finiteintersection propertywhen the following holds: LetF be a family of closed sets of X with$\cap\{F : F \in \mathrm{F}\} = \emptyset$, then there exists a finite subfamily $F_1, F_2, \ldots F_n$ of elements of F such that$\bigcap_{i=1}^{n} \emptyset_i$.we will use a different, but equivalent, form of the finite intersection property: if, for all finite subfamilies of F we have$\bigcap_{i=1}^{n} \emptyset_i$, then $\cap\{F : F \in \mathrm{F}\} \neq \emptyset$. A topological space is compact if and only if it has the finite intersection property.A topological group is a set that has both a topological structure and an algebraic structure. We consider a metric space is a generalization of a Euclidean space and a topological space is a generalization of a metric space. Instead of having a metric that tells us the distance between two points, topological spaces rely on a different notion of closeness; points are related by open sets.

The Properties of compactness in topological spaces are defined (X, τ) be a topological space and $S \subset X$. *An* open cover O *of* S is a collection of open sets that contain S, in symbols we have $S \subset \cup\{U : U \in O\}$.Let$(X, \tau)$ be a topological space and $S \subset X$. The set S is compact *in* (X, τ) when every open cover has a finite sub cover. That is, for any collection O that covers S, there exist $U_1, U_2, \ldots U_n \in O$ such that $S \subset U_{i=1}^{n} U_i$.

2 Theorem

Let(X, τ_X) and(Y, τ_Y)be a topological spaces and let $f : X \to Y$ be a function. The following three statements are equivalent.

i) $f : X \to Y$ *is* continuous *on* X,

ii) $f^{-1}(U)$ is open in X for all open sets U *in* Y ,

iii) $f^{-1}(F)$ is closed in X for all closed sets F *of* Y .

Many statements that are true for continuous functions in metric spaces are also true in topological spaces. An example of this is the transitivity of continuity.

3 Theorem

Let(X, τ_X) and$(Y, \tau_Y)$and(Z, τ_Z) be topological spaces such that $g : X \to Y$ *is* continuous *at* $x_0 \in X$ and $f : Y \to Z$ is continuous at $g(x_0) \in Y$. It follows that$(f \circ g)(x)$ is continuous at x_0.

Proof

By using definition, we have: $g : X \to Y$ is continuous at x_0: $N \in N_{g(x0)}$ implies $g^{-1}(N) \in N_{x0}$ $f : Y \to Z$ is continuous at $g(x_0)$: $M \in N_{f(g(x0))}$ implies $f^{-1}(M) \in N_{g(x0)}$. In order to prove that $(f \circ g)(x)$ is continuous at x_0 it must be shown that $N \in N_{(f \circ g)(x0)}$ implies $g^{-1}(f^{-1}(N)) \in N_{x0}$. Let us now assume that $N \in N_{(f(g(x0))}$. Since f is continuous at $g(x_0)$, it follows that $f^{-1}(N) \in N_{g(x0)}$. Since g is continuous at x_0 we have $g^{-1}(f^{-1}(N)) \in Nx0$, as required. Hence, it is proved. Wediscussconvergence in a topological space for which let us define what a net is, and its relevant properties.

4 Theorem

Let(X, τ) be a topological space and S a subset of X. Theclosure*of* S is the setof all points in X that are a limit of a net in S. This can be worded as: x $\in \overline{S}$ if and only if x is the limit of a net in S. This is a property of limits in metric spaces that carries over to topological spaces.

Proof

Let us consider N_x as a directed set defined by M $\preccurlyeq$ Nwhen $N \subset M$ for $M, N \in N_x$ for some x. Assume that $x \in \overline{S}$, by definition (iii) we find that for all $N \in N_x$, there exists an $x_N \in N \cap S$. We thus find that the net $(x_N)_{N \in N_x}$ converges to x, this is because all neighbourhoods of x contain x. For the other direction let $(x_\alpha)_{\alpha \in A}$ be a net in S that converges to x. Let us assume by way of contradiction that $x \notin \overline{S}$, so $x \in X \setminus \overline{S}$. We know that $\overline{S}$ is closed, this implies that $X \setminus \overline{S}$ is open, which means that $X \setminus \overline{S}$ is a neighbourhood of x. Since $(x_\alpha)_{\alpha \in A}$ converges to a point outside of $\overline{S}$, there exists some $\alpha_N \in A$ such that $x_\alpha \in X \setminus \overline{S}$ for all $\alpha_N \preccurlyeq \alpha$. We know hat $x \setminus \overline{S} \subset x \setminus S$, so $x_\alpha \in X \setminus S$ for all $\alpha_N \preccurlyeq \alpha$. This is a contradiction since $(x_\alpha)_{\alpha \in A}$ is a net in S. Hence, the theorem is proved.

5 Theorem

Let(X, τ_1) be a compact space, let(Y, τ_2) be a topological space and let$f : X \to Y$ be a continuous function. Then $f(X)$ is compact.

Proof

Let O be an open over for $f(X)$. By using theorem (6) we know that $\{f^{-1}(U) : U \in O\}$ is an open cover for X. Since X is compact, there is a finite subcover. There exist U_1, U_2, $\ldots U_n \in O$ such that $X \subset f^{-1}(U_1) \cup f^{-1}(U_2) \cup \ldots \cup f^{-1}(U_n)$. It follows that $f(X) \subset U_1 \cup U_2 \cup \ldots \cup U_n$. Thus, an arbitrary open cover of $f(X)$ has a finite subcover. Hence, the theorem is proof.

6 Theorem

Let(X, τ) be a compact topological space and let Y be a closed subset of X. Then Y is a compact set.

Proof

Let O be an open cover for Y. Because Y is closed in X, X/Y is open in X. From this we get that $O \cup X/Y$ is an open cover for X. Since X is compact there must be a finite subcover. So, there exist U_1, U_2, $\ldots U_n \in U$ such that $X \subset U_1 \cup U_2 \cup \ldots \cup U_n \cup (X/Y)$. It follows that $Y \subset U_1 \cup U_2 \cup \ldots \cup U_n$. An arbitrary open cover of Y has a finite subcover, thus Y is compact.

7 Theorem

Let(X, τ) be a topological space. If K_1, K_2, ... K_n are a family of compact sets in(X, τ), then $U_{i=1}^{n} K_i$is compact.

Proof

Let O be an open cover for $U_{i=1}^{n} K_i$. Since K_i is a subset of $U_{i=1}^{n} K_i$for all i, then O is an open cover for all K_i. Since each K_i is compact, there is a finite subcover of O for each i, we denote each subcover by O_i. It follows that $U_{i=1}^{n} Q_i$ covers$U_{i=1}^{n} K_i$. The finite union of a collection of finite sets in finite and $U_{i=1}^{n} O_i$ so we have our finite subcover. This completes the proof.

8. References

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