Study On the Application And Principles of Functional Analysis

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Abstract-
Functional analysis is a branch of mathematics that focuses on the study of vector spaces and functions that exist within those spaces. The primary objective of this field is to get a knowledge of the characteristics and way these functions behave. Applications of this topic may be found in a broad variety of fields, including mathematics, physics, engineering, and computer science. The purpose of this abstract is to investigate the many applications of functional analysis in a variety of fields, focusing on the role that it plays in resolving issues that occur in the real world and improving scientific knowledge. When it comes to quantum physics, where systems often display behaviour that is infinite-dimensional, functional analysis is a very important technique since it gives strong tools for investigating spaces that have an unlimited number of dimensions. In the field of quantum mechanics, the theory of operators and Hilbert spaces plays an essential part, since it makes it easier to formulate and analyse quantum states and observables.

1. INTRODUCTION
1.1 NORMED AND BANACH SPACES
1.1 Vector spaces
Here, we will review the definition of a vector space that was presented before. In a general sense, it is a collection of items that are referred to as "vectors." The addition of any two vectors may produce a new vector, and any vector can be multiplied by an element from R (or C, depending on whether we are considering a real or complex vector space) to produce a new vector. Both operations can be performed to produce a new vector. In the following, the exact definition is provided.

Definition. Let K = R or C (or more generally1 a field). A vector space over K, is a set X together with two functions, $+ : X \times X \to X$, called vector addition, and $\cdot : K \times X \to X$, called scalar multiplication that satisfy the following:

V1. For all $x_1, x_2, x_3 \in X$, $x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$.

V2. There exists a component, which is indicated by the number 0 and is referred to as the zero vector, in which for all $x \in X, x + 0 = 0 + x = x$.

V3. For every $x \in X$, there exists an element, denoted by $-x$, such that

$$x + (-x) = (-x) + x = 0.$$

V4. For all $x_1, x_2 \in X$, $x_1 + x_2 = x_2 + x_1$.

V5. For all $x \in X, 1 \cdot x = x$.

V6. For all $x \in X$ and all $\alpha, \beta \in K, \alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$.

V7. For all $x \in X$ and all $\alpha, \beta \in K, (\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$.

V8. For all $x_1, x_2 \in X$ and all $\alpha \in K, \alpha \cdot (x_1 + x_2) = \alpha \cdot x_1 + \alpha \cdot x_2$.

Examples.

1. R is a vector space over R, and the addition of vectors is the same as the addition of real numbers, while the multiplication of real numbers is the same as the multiplication of scalars.

2. The following is a definition of addition and scalar multiplication in the vector space Rn, which is a vector space over R:

$$\text{If } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n, \text{ then } \alpha \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

3. The sequence space $\ell\infty$ is given. The idea of a vector space is remarkably comprehensive, as shown by this example and the one that follows it. This is the initial impression that is given.

The vector space of all bounded sequences with values in K is denoted by the symbol $\ell\infty$. The vector space is associated with addition and scalar multiplication, which are defined as follows:

$$(x_n)_{n\in\mathbb{N}} + (y_n)_{n\in\mathbb{N}} = (x_n + y_n)_{n\in\mathbb{N}}\ , (x_n)_{n\in\mathbb{N}}\ , (y_n)_{n\in\mathbb{N}} \in \ell^\infty;$$
$$\alpha(x_n)_{n\in\mathbb{N}} = (\alpha x_n)_{n\in\mathbb{N}}\ , \alpha \in \mathbb{K}\ , (x_n)_{n\in\mathbb{N}} \in \ell^\infty;$$

4. The function space $C[a, b]$. Let $a, b \in$ Rand $a < b$. Consider the vector space comprising functions $f : [a, b] \rightarrow K$ that are continuous on $[a, b]$, with addition and scalar multiplication defined as follows.

If $f, g \in C[a, b], \alpha \in K$ then $f + g \in C[a, b], \alpha f \in C[a, b]$ are the functions given by

$$(f + g)(x) = f(x) + g(x), \qquad x \in [a, b]$$
$$(\alpha f)(x) = \alpha f(x)\ , \qquad x \in [a, b]$$

$C[a, b]$is referred to as a ‘function space’, since each vector in $C[a, b]$ is a function
$(C: [a, b] \rightarrow K)$.

1.2 Normed spaces

In order to perform "calculus" in vector spaces, which means to discuss limiting processes, convergence, approximation, and continuity, we need a concept of "distance" or "closeness" between the vectors that make up the vector space. The concept of a norm is used to offer this information.

Definitions. Let X be a vector space over R or C. A norm on X is a function k· k : X → [0, +∞) such that:

N1. (Positive definiteness) For all $x \in X, \|x\| \geq 0$. If $x \in X$, then $\|x\| = 0$ iff $x = 0$.

N2. For all $\alpha \in R$ (respectively C) and for all $x \in X, \|\alpha x\| = |\alpha|\|x\|$.

N3. (Triangle inequality) For all $x, y \in X, \|x + y\| \leq \|x\| + \|y\|$.

A normed space is a vector space X equipped with a norm.

If $x, y \in X$, then the number $\|x - y\|$ provides a notion of closeness of points x and y in X, that is, a ‘distance’ between them. Thus $kxk = kx - 0k$ is the distance of x from the zero vector in X.We will now provide a few instances of spaces that are normed.

1.3 Banach spaces: A Banach space is a complete normed space $(X, \|\cdot\|)$. A normed space is a order pair $(X, \|\cdot\|)$ consisting of a vector space X over a scalar field $\mathbb{K}$(where$\mathbb{K}$ is commonly $\mathbb{R}$ or $\mathbb{C}$) together with a distinguished norm $\|\cdot\|: X \longrightarrow \mathbb{R}$.

The norm$\|\cdot\|$ of a normed space $(X, \|\cdot\|)$ is called a complete norm if $(X, \|\cdot\|)$ is a Banach space.

2. FUNDAMENTAL THEOREMS OF FUNCTIONAL ANALYSIS

(Baire’s theorem, the open mapping theorem and the closed graph theorem)

2.1 Baire’s theorem

Statement: Let X be a complete metric space If$\{G_n\}_{n\geq 1}$ is a sequence of dense and open subsets of X, then$A = \bigcap_{n=1}^{\infty} G_n$ is also dense.

Proof. A is dense in X if and only if $\overline{A} = X$, that is, for all $x \in X$ and all $r > 0, B(x, r) \cap A \neq \emptyset$. This is equivalent to prove that$A \cap G \neq \emptyset$ for every nonempty open set G in X.

SinceG_1 is dense and open,$G_1 \cap G$ is a nonempty open set. Therefore, there exista$_1 \in G_1 \cap G$ and $r_1 > 0$ so that$\overline{B(a_1, r_1)} \subset G_1 \cap G$.Similarly, $G_2$is dense and open, so there exista$_2 \in G_2 \cap B(a_1, r_1)$ and $0 < r_2 < \frac{r_1}{2}$ so that$\overline{B(a_2, r_2)} \subset G_2 \cap B(a_1, r_1)$. By induction, we can build the sequences$\{a_n\}_{n\geq 1} \subset X$ and $\{r_n\}_{n\geq 1} \subset \mathbb{R}^+$ with$0 < r_{n+1} < \frac{r_n}{2} = \frac{r_1}{2^n}$ and $\overline{B(a_{n+1}, r_{n+1})} \subset G_{n+1} \cap B(a_n, r_n)$.

Besides, $\{a_n\}_{n\geq 1}$ is a Cauchy sequence in X. Indeed, given $n, m \in$ Nwith $m < n$, then $a_n \in \overline{B(a_m, r_m)}$ and$d(a_n, a_m) \leq r_m < \frac{r_1}{2^{m-1}} \rightarrow 0$ as $n, m \rightarrow \infty$. Since X is complete, there exists $a = \lim_{n\to\infty} a_n$in X.

Finally, it is readily shown that $a \in A \cap G$, that is, $a \in G \cap G_m$ for all $m \in N$.

Indeed, $a_n \in \overline{B(a_m, r_m)}$ whenever $n \geq m$, together with $a = \lim_{n\to\infty} a_n$ implies that $a_n \in \overline{B(a_m, r_m} \subset G_m$ for all $m \in N$. Besides, $a \in \overline{B(a, r_1)} \subset G$ and, hence, $A \cap G \neq \emptyset$.
Corollary 2.1.1. Let $X = \bigcup_{n=1}^{\infty} F_n, F_n$ be a complete metric space and $\{F_n, n \in N\}$ a sequence of closed sets in X. Then, there is one F_n with nonempty interior.
Proof. Since $X = \bigcup_{n=1}^{\infty} F_n, \emptyset = X^c = \bigcap_{n=1}^{\infty} F_n^c$ where the sets F_n^c are open.
Baire's theorem states that there is at least one F_n^c not dense.
Thus, $\overline{F_n^c} \neq X$ and consequently, $\frac{X}{\overline{F_n^c}} = \operatorname{int}(F_n) \neq \emptyset$.

2.2 The open mapping theorem

Definition 2.2.1. A linear operator $T: E \to F$ is said to be open if $T(G)$ is an open set in F for any open set G in E.
Theorem 2.2.2 (Open mapping theorem). Let E, F be two Banach spaces and $T: E \to F$ a linear operator that is surjective and continuous. Then, T is a mapping that is open.
Proof. We want to prove that $T(G)$ is an open set in F for any open set G in E.
1. It is enough to prove that $T(B(0, r))$ is a neighbourhood of zero in F for all $r > 0$. Let $G \subset E$ be an open set. Since T is surjective, we consider $T a \in T(G)$ with $a \in G$. Since G is open, there is $r > 0$ so that $B(a, r) = a + B(0, r) \subset G$. By linearity, $T(B(a, r)) = T a + T(B(0, r)) \subset T(G)$. The hypothesis assures that $T(B(0, r))$ is a neighbourhood of zero, so $T(B(a, r))$ is a neighbourhood of Ta in F. Hence, $T(G)$ is open.
2. For all $r > 0, \overline{T(B(0, r))}$ is a neighbourhood of zero in F, that is, there is $\sigma > 0$ so that $B(0, \sigma) \subset \overline{T(B(0, r))}$
Consider the following expressions,
$E = \bigcup_{n=1}^{\infty} B\left(0, \frac{nr}{2}\right)$ and $F = T(E) = T\left(\bigcup_{n=1}^{\infty} B\left(0, \frac{nr}{2}\right)\right) = \bigcup_{n=1}^{\infty} T\left(B\left(0, \frac{nr}{2}\right)\right)$
Note that $F \subset \bigcup_{n=1}^{\infty} \overline{T\left(B\left(0, \frac{nr}{2}\right)\right)} \subseteq \overline{\bigcup_{n=1}^{\infty} T\left(B\left(0, \frac{nr}{2}\right)\right)} = \overline{F} = F$.
Hence $F = \bigcup_{n=1}^{\infty} \overline{T\left(B\left(0, \frac{nr}{2}\right)\right)}$
By Corollary 2.1.1, there is $N \in N$ such that $nt\left(\overline{T\left(B\left(0, \frac{nr}{2}\right)\right)}\right) \neq \emptyset$.
We can assume $N = 1$ because $\overline{T\left(B\left(0, \frac{Nr}{2}\right)\right)} = N \cdot \overline{T\left(B\left(0, \frac{r}{2}\right)\right)} \cong \overline{T\left(B\left(0, \frac{r}{2}\right)\right)}$.
Hence, there exist $y \in F$ and $\sigma > 0$ so that $B(y, \sigma) = y + B(0, \sigma) \subseteq \overline{T(B(0, r/2))}$.
Besides, there exists a sequence $\{x_n\}_n \subset B\left(0, \frac{r}{2}\right)$ such that,
$y = \lim_n Tx_n$ and $-y = \lim_n T(-x_n)$. Therefore, $-y \in \overline{T\left(B\left(0, \frac{r}{2}\right)\right)}$.
Finally, we have that

$$B(0, \sigma) \subseteq -y + \overline{T\left(B\left(0, \frac{r}{2}\right)\right)} \subseteq \overline{T\left(B\left(0, \frac{r}{2}\right)\right)} + \overline{T\left(B\left(0, \frac{r}{2}\right)\right)} \subseteq \overline{T(B(0, r))}$$

3. Fixed $s > 0, T(B(0, s))$ is a neighbourhood of zero in F.
We write $s = \sum_{n=1}^{\infty} r_n$ with $r_n > 0$ (obviously, $r_n \to 0$ as $n \to \infty$). According to the second step of this proof, for all $n \geq 1$ there exists $\sigma_n > 0$ such that $B(0, \sigma_n) \subset \overline{T(B(0, r_n))}$. We can assume that $\sigma_n \to 0$.
Let $y \in (0, \sigma_1) \subset \overline{T(B(0, r_1))}$. Since T is surjective, there exists $x_1 \in B(0, r_1)$ so that $\|y - Tx_1\|_F < \sigma_2$.
It follows that $y - Tx_1 \in (0, \sigma_2) \subset \overline{T(B(0, r_2))}$. Then, there exists $x_2 \in B(0, r_2)$ so that $\|y - Tx_1 - Tx_2\|_F < \sigma_3$.
By induction, if $y - Tx_1 - \cdots - Tx_{n-1} \in (0, \sigma_n) \subset \overline{T(B(0, r_n))}$,
then there exists $x_n \in B(0, r_n)$ so that $\|y - Tx_1 - \cdots - Tx_{n-1} - Tx_n\|_F < \sigma_{n+1}$.
Since E is a Banach space and $\sum_{n=1}^{\infty} \|x_n\|_E < \sum_{n=1}^{\infty} r_n = s < \infty$

there exists $x = \sum_{n=1}^{\infty} r_n \in E$. Note that $\|x\|_E \le \sum_{n=1}^{\infty}\|x_n\|_E < s$ implies that $x \in B(0,s)$. Since T is continuous, $y = \lim_{n\to\infty} T(\sum_{k=1}^{n} x_k) = Tx \in T\big(B(0,s)\big)$

Hence, $B(0,\sigma_1) \subset T(B(0,s))$ and $T(B(0,s))$ is a neighbourhood of zero.

Corollary 2.2.3 (Banach isomorphism theorem). Assume that E and F are two Banach spaces, and that T is a bijective continuous linear operator that maps Eto F. Further, it may be said that T^{-1} is likewise a bijective continuous linear operator. T is an example of an isomorphism.

Proof. The open mapping theorem states that T is an open set. Given that T is both bijective and open, it is possible to get T^{-1}, which is a continuous function.

2.3 Closed Graph Theorem:

2.3.1 **Definition:** Let E, F be normed spaces. Then the linear operator $T: E \to F$ is said to be closed operator if for every sequence $\{x_n\}$ in E such that

$x_n \to x$ and $Tx_n \to y \Longrightarrow Tx = y$

Alternative Definition: Define a normed space $E \times F$, where the two algebraic operations are defined as,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$\alpha(x, y) = (\alpha x, \alpha y),$$

And the norm on $E \times F$ is defined by

$$\|(x, y)\| = \|x\| + \|y\|.$$

Then the operator $T: E \to F$ is closed operator if the graph of T,

$G(T) = \{(x, Tx): x \in E\}$ is closed in $E \times F$

Remark: Continuous linear operator is closed linear operator but the converse is not true (under certain conditions the converse is true which is stated in following theorem.

2.3.2 Theorem: (Closed Graph theorem): Let E, F be two Banach spaces and $T: E \to F$ is a linear operator, then T is continuous $\Longleftrightarrow$ T is closed

Proof. If T is continuous linear operator, then obviously T is closed linear operator. It sufficiently prove the converse, Conversely, suppose T is closed operator. Then the graph of $T, G(T)$ is closed in $E \times F$. Moreover, it is subspace and so it is a complete space. Define $P: G(T) \to E$ by $P(x, Tx) = x$. It is easy to verify that P is continuous and surjective. By bounded inverse theorem, $P^{-1}: E \longrightarrow G(T)$ is continuous, that is $\|P^{-1}(x)\| \le c\|x\|, \forall x \in E$ for some $c > 0$.

Hence T is bounded because of $\|T(x)\| \le \|Tx\| + \|x\| = \|(x, Tx)\| = \|P^{-1}(x)\| \le c\|x\|, \forall x \in E$

2.3.3 Definition (Perfectly convex set). A set K in a Banach space, Y is called perfectly convex if for every sequence $\{x_k\}_{k=1}^{\infty}$ and every numbers $\lambda_k \ge 0$ such that $\sum_{k=1}^{\infty}\lambda_k = 1$, one has $\sum_{k=1}^{\infty}\lambda_k x_k \in K$

It is only for finite sequences $\{x_k\}$ that convex sets are able to meet this feature. (Why is that?) As a result, any set that is absolutely convex is also convex, but the opposite is not true. Here is an illustration:

Proof. Assume $B = \epsilon B_Y \subseteq \overline{K}$ we would like to show that $\frac{1}{2}B \subseteq K$ .The assumption clearly implies that $B \subseteq K + \frac{1}{2}B$,

for the right side is the $\in$ −neighbourhood of K in Y. Iterating this inclusion gives

$$B \subseteq K + \frac{1}{2}\left(K + \frac{1}{2}B\right) = K + \frac{1}{2}K + \frac{1}{4}B$$
$$\subseteq K + \frac{1}{2}K + \frac{1}{4}\left(K + \frac{1}{2}B\right) = K + \frac{1}{2}K + \frac{1}{4}K + \frac{1}{8}B \subseteq \cdots$$

Therefore, $B \subseteq K + \frac{1}{2}K + \frac{1}{4}K + \frac{1}{8}K + \cdots$

By perfect convexity (check!), we have $\frac{1}{2}B \subseteq \frac{1}{2}K + \frac{1}{4}K + \frac{1}{8}K + \frac{1}{16}K + \cdots \subseteq K$

This proves the lemma.

3. CONCLUSION

In conclusion, functional analysis serves as a foundational framework with wide-ranging applications across numerous scientific and technical fields. To solving difficult issues in quantum physics, engineering, signal processing, and data analysis, it is vital to have tools that are capable of handling infinite-dimensional spaces, studying operators, and analysing functions. In addition to contributing to innovation and development in scientific and technical areas, the continual improvements in functional analysis continue to influence our comprehension of the mathematical foundations that underlie a variety of disciplines.

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